MATH 3310 Assignment 4

Due: November 20, 2024

1. QR factorization gives a sequence of matrices $\{A^{(0)}, A^{(1)}, A^{(2)}, \ldots\}$, where

$$A^{(0)} = \begin{pmatrix} 1 & 2 & 9 \\ 0 & 2 & 1 \\ 1 & 2 & -3 \end{pmatrix}$$

   Find the QR factorization of $A^{(0)}$ by Gram-Schmidt process. Also compute $A^{(1)}$. Please show all your steps.

2. Let A be a non-singular $n \times n$ real matrix. We apply the QR method on A to obtain a sequence of matrices $\{A^{(j)}\}_{j=0}^{\infty}$, which satisfies:

$$\begin{aligned} A^{(0)} &= A; \\ A^{(j+1)} &= R^{(j)} Q^{(j)} \text{ for } j = 0, 1, 2, ..., \end{aligned}$$

   where $A^{(j)} = Q^{(j)} R^{(j)}$ is the QR factorization of $A^{(j)}$. Let k be an integer greater than 2020. Given that the QR factorizations of A^{k-1} and $A^{(k-1)}$ are given by

$$A^{k-1} = Q_1 R_1 \text{ and } A^{(k-1)} = Q_2 R_2.$$

   In this question, all QR factorization is obtained in such a way that the diagonal entries of the upper triangular matrix are positive.

   (a) Express A^2 in terms of $Q^{(0)}$, $Q^{(1)}$, $R(0)$, and $R^{(1)}$, and show that A^k can be expressed in terms of $Q^{(0)}, \ldots, Q^{(k-1)}$ and $R^{(0)}, \ldots, R^{(k-1)}$. Please explain your answer in detail.

   (b) Express A in terms of Q_1, Q_2, R_1 and R_2 only. Please explain your answer in detail.

   (c) Starting from $\mathbf{x}_0$, we apply the Power's method on A as follows:

$$\mathbf{x}_{j+1} = \frac{A\mathbf{x}_j}{||A\mathbf{x}_j||_\infty} \text{ for } j = 0, 1, 2, ... \,.$$

   Write $\mathbf{x}_k$ in terms of $\mathbf{x}_0$, Q_1, Q_2, R_1 and R_2 only (without A and k). Please explain your answer in detail.

3. Suppose A is a real symmetric matrix and $A = QDQ^T$. Assuming that the diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$ of D are arranged in descending order in terms of their magnitudes and satisfy

$$|\lambda_1| = |\lambda_2| = ... = |\lambda_k| > |\lambda_{k+1}| \geq ... \geq |\lambda_n|,$$

where $k < n$. Denote the j-th column of Q by $\mathbf{q}_j$, where $\mathbf{q}_1$, $\mathbf{q}_2$, ..., $\mathbf{q}_n$ are eigenvectors of A associated to λ_1, λ_2,..., λ_n respectively.

Let $\mathbf{x}_0$ be the initial vector defined as $\mathbf{x}_0 = a_1\mathbf{q}_1 + a_2\mathbf{q}_2 + ... + a_n\mathbf{q}_n$, where $a_j \in \mathbb{R}$ for $1 \leq j \leq n$ and $a_i \neq 0$ for $i = 1, 2, ..., k$. Consider the iterative scheme:

$$\mathbf{x}_{j+1} = \frac{A\mathbf{x}_j}{||A\mathbf{x}_j||_\infty} \text{ for } j = 0, 1, 2, ...$$

(a) Suppose $\lambda_1 = \lambda_2 = ... = \lambda_k \in \mathbb{R}$. will $||A\mathbf{x}_j||_\infty$ always converge as $j \to \infty$. If yes, what will it converge to? If not, please provide a counter-example and explain your answer in detail. Please show the full details of your proof.

(b) In general, if $|\lambda_1| = |\lambda_2| = ... = |\lambda_k|$, will $||A\mathbf{x}_j||_\infty$ always converge $j \to \infty$? If yes, what will it converge to? If not, please provide a counter-example and explain your answer in detail. Please show the full details of your proof.

4. Let A be an $n \times n$ complex matrix, whose eigenvalues satisfy:

$$|\lambda_1| > |\lambda_2| > |\lambda_3| > \cdots > |\lambda_n| > 0$$

Also, we define the following:

$$\begin{aligned} \cos\angle(x, y) &= \frac{|\langle x, y\rangle|}{\|x\|\|y\|}; \\ \sin\angle(x, y) &= \sqrt{1 - \cos^2\angle(x, y)}; \\ \tan\angle(x, y) &= \frac{\sin\angle(x, y)}{\cos\angle(x, y)}. \end{aligned}$$

If $\cos\angle(x, y) = 0$, then let $\tan\angle(x, y) = \infty$. Here, $\langle x, y\rangle = \sum_{i=1}^n x_i\bar{y}_i$, where x_i and y_i are the i-th entries of $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^n$ respectively.

(a) Suppose $A = QDQ^*$ where Q is unitary (i.e. $Q^*Q = I$, where Q^* is the conjugate transpose of Q) and D is diagonal, prove that

$$\cos\angle(Q^*x, Q^*y) = \cos\angle(x, y).$$

(b) Consider the power method in the form

$$x^{(n)} = Ax^{(n-1)}$$

Assume that $\cos \angle(x^{(0)}, e_1) \neq 0$, where e_1 is the eigenvector associated to the dominant eigenvalue λ_1. Prove that

$$\tan \angle(x^{(m+1)}, e_1) \leq \frac{|\lambda_2|}{|\lambda_1|} \tan \angle(x^{(m)}, e_1).$$

(c) For some $\mu \in \mathbb{R}$, let $A - \mu I$ be invertible. Assume

$$|\lambda_1 - \mu| < |\lambda_2 - \mu| \leq \cdots \leq |\lambda_n - \mu|.$$

Under the same notations and assumptions in (b), consider the shifted inverse power iteration

$$x^{(m)} = (A - \mu I)^{-1} x^{(m-1)}.$$

Using part (b) or otherwise, prove that

$$\tan \angle(x^{(m+1)}, e_1) \leq \frac{|\lambda_1 - \mu|}{|\lambda_2 - \mu|} \tan \angle(x^{(m)}, e_1).$$

(d) Suppose A is a real matrix, and for $x \in \mathbb{R}^n \setminus \{0\}$, define the Rayleigh quotient $R(x, A) = \frac{x^* A x}{x^* x}$. Let r be a nonzero eigenvector of A for the eigenvalue λ, show that

$$|R(x, A) - \lambda| \leq \rho(A - \lambda I) \sin \angle(x, r) \leq \rho(A - \lambda I) \tan \angle(x, r)$$

(Hint: show that $\sin \angle(x, y) = \min \left\{ \frac{\|x - \alpha y\|}{\|x\|} : \alpha \in \mathbb{R} \right\}$ and note that $(A - \lambda I)r = 0$)